

# Relational Measurements

and

## Uncertainty

by

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### *Abstract:*

*In representational measurement theory, the current theory of all measurements, calibration and sampling processes are assumed to be a linear transformation of the coordinate system, of no effect. In this paper calibration and sampling are shown to be independent non-linear processes which do change measurement results. Relational measurement theory is developed to include calibration and sampling. The measurement changes caused by calibration and sampling are proven to be equal to the quantum measurement disturbance described by the universal uncertainty relation which has been verified by experiments. Therefore relational measurement theory explains the measurement disturbance in quantum mechanics.*

### Keywords:

Measurement disturbance, measurement model, calibration, sampling, references

## 1. INTRODUCTION

In representational measurement theory [1] a measurement result is a magnitude of equal intervals. In this paper relational measurement theory [2] defines a *measurement result* as a sum of intervals, where each interval's magnitude is modified by both calibration and sampling processes. The application of relational measurement theory to quantum systems explains the disturbance of one observable when measuring a second observable [3]. In this paper: *sampling* is the division of a continuous observable into discrete intervals [4], a *measurement* is the sum of the projection of all these discrete intervals onto the measuring apparatus intervals [5], and *calibration* is the correlation of sets of the measuring apparatus intervals to externally defined intervals [6]; together these three processes produce a measurement result.

In von Neumann's development of a measurement in Hilbert space [7], the possible discrete elements are equivalent (not necessarily equal) and of varying probability. Each such quantum element (in one dimension) may be seen as one of a measuring apparatus' minimum intervals in the same dimension. In representational measurement theory, these elements/intervals are assumed to be equal. Then the standard deviation of a distribution of these elements/intervals is zero.

The Heisenberg Uncertainty Relation (HUR) [8] identifies an inconsistency when the elements/intervals of a measurement system are assumed to be equal. In this paper the standard deviations in the HUR, shown in equation (7), are proved to be of the magnitudes of the observable's intervals, not of the magnitude of the observable, and are never zero. In the relational measurement view of the HUR, each observable's standard deviation is a measure of a distribution of calibration corrections. These calibration corrections appear as a measurement disturbance of the second observable when both observables are correlated by calibration corrections to the same reference.

Sections 2 – 5 develop relational measurement theory by applying concepts from classical metrology. Sections 6 – 7 apply relational measurement theory to the HUR, proving that the standard deviation of a distribution of discrete intervals in the HUR equals the effect of calibration on experimental measurement results.

In 2003, Ozawa [3] developed the universal uncertainty relation, equation (12), which is shown to support relational measurement theory. His universal uncertainty relation, with a minor modification, formalizes both calibration and sampling indeterminacy [9]. In the Appendix, experiments by others [10] are presented which verify the universal uncertainty relation and therefore relational measurement theory.

## 2. RELATIONAL MEASUREMENTS

A *relational measurement system* defines the *mean measurement result* as the product of measurement magnitude and the standard deviation of the measurement intervals due to calibration and sampling processes (4).

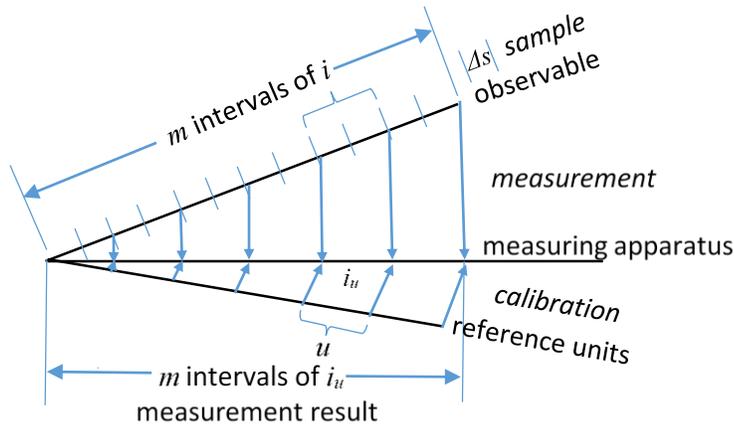


Fig. 1 Relational measurement system

Fig. 1 identifies  $m$  intervals of  $i_u$ , a relational measurement result, relative to  $m$  intervals of  $i$ , a representational measurement. Sampling, often treated as part of a measurement process, divides the observable into increments ( $\Delta s$ ) indicated in Fig. 1 by short lines at right angles to the observable. A measurement is indicated by downward arrows which project the intervals (usually two or more sample increments) onto the measuring apparatus to be summed. Calibration of the measuring apparatus intervals to  $u$  (a reference unit) is indicated by the inward arrows. Calibration is represented in an orthogonal plane as it is independent of sampling and measurement. Both planes have a common basis, the measuring apparatus.

Example:  $X$  is a rod's length (the observable).  $X$  is experimentally defined as a magnitude ( $m$ ) of intervals each correlated to a minimum reference unit  $x$  (in this example a centimeter). The magnitude of each correlated interval is  $i_x$ . The representational (normalized) magnitude of the intervals of  $X$  is  $i = 1/m$ . The experimental measurement result precision [11] (local variation) is  $\pm \Delta s_{xi}$  per interval  $i$  [12]. The measuring apparatus is a meter scale which defines the magnitude of the intervals, where each interval is calibrated to  $x$ , the reference interval. A calibration process makes feasible comparisons between independent measurement results. The measured accuracy [11] (relative variation) of each  $i_x$  is determined by calibration to  $x$ . The differences between  $mi$ ,  $mx$  and  $mi_x$  are not treated in representational measurement theory.

Applying Fig. 1,  $\Delta x_i$  = calibration variation of each interval and  $\Delta s_{xi}$  = sample increment of each  $i$  interval, the calibration (1) and sampling (2) operators are:

$$x \pm \Delta x_i = i_x \quad (1)$$

$$i \pm \Delta s_{xi} = i_x \quad (2)$$

The indeterminacy (the sum of calibration and sampling variation) of  $i_x$  is:

$$\Delta i_x = (\mp \Delta x_i) + (\pm \Delta s_{xi}) \quad (3)$$

In representational measurement theory, this indeterminacy is assumed to reduce to zero as accuracy and precision move towards perfect [13]. This is not experimentally possible. In all

discrete measurements of continuous observables there is a non-cancelable minimum interval indeterminacy  $\min\Delta i_x <$  the product of the speed of light and the reciprocal of the highest sampling frequency observed. The higher the sampling frequency, the smaller  $|\min\Delta i_x|$  (the vertical bars represent an absolute value). When all other indeterminacy is nulled, the magnitude of each interval  $i_x$  randomly deviates by just less than  $\pm \Delta_{Sxi}$ , the sampling increment. Therefore  $\Delta x_i$  cannot completely cancel  $\Delta_{Sxi}$ .

Equation (4) presents four different functions which represent the observable  $\langle X \rangle$  (where brackets indicate the mean of  $X$ ) based upon three different assumptions.

$$\int \psi^* \hat{X} \psi dx \rightarrow \sum_{i=1}^{i=m} i \rightarrow \sum_{i_x=1}^{i_x=m} i_x \cong m [\sigma(i_x) + \sigma(\Delta_{Sxi})] = \langle X \rangle \quad (4)$$

The first function is from quantum mechanics [14]:  $\int \psi^* \hat{X} \psi dx$ .  $\psi^*$  is the complex conjugate of the state vector  $\psi$  (observable) of  $x$  and  $\hat{X}$  is the operator of  $X$ , therefore  $\int \psi^* \hat{X} \psi dx$  (assumes infinitesimal sampling increments) represents the mean of a continuous observable. Normalized sampling (assumes  $i = x$ ) transforms  $\int \psi^* \hat{X} \psi dx$  to a discrete measurement [15],

shown as the second function:  $\sum_{i=1}^{i=m} i$  which is the rod's length,  $m$  intervals of  $i$ .

Calibration (1) transforms  $\sum_{i=1}^{i=m} i$  to the third function:  $\sum_{i_x=1}^{i_x=m} i_x$  which is the rod's length in  $i_x$  intervals and is the current classical metrology model [16]. When the sampled increment  $\Delta_{Sxi} \ll i_x$  (assumed in classical metrology),  $\sum_{i_x=1}^{i_x=m} i_x$  is close to the fourth function:  $m [\sigma(i_x) + \sigma(\Delta_{Sxi})]$  as  $\sigma(\Delta_{Sxi}) < I$  sampled increment (derived below in Section 3).

The fourth function,  $m [\sigma(i_x) + \sigma(\Delta_{Sxi})]$ , does not require these three assumptions. This function sums over the common basis both the calibration (1) and sampling (2) operators and is the only function of the four which represents the mean of the experimental measurement results of a continuous observable at all experimentally possible sampling frequencies.

### 3. SAMPLING AND CALIBRATION EXAMPLES

Consider a digital voltmeter (measuring apparatus) where  $00.01$  is the voltmeter's display of the minimum interval. Measuring a fixed voltage (observable) multiple times produces a stochastic distribution of measurement results. The maximum indeterminacy of this distribution is specified by the manufacturer, for all voltmeters of this model, to be  $\pm 1\%$  indeterminacy. To maintain the  $\pm 1\%$  of a  $0.01$  volt measurement requires  $\pm 0.0001$  volt precision, i.e., each interval is between  $0.0099$  and  $0.0101$  volts. The  $\pm 1\%$  indeterminacy allows a laboratory with multiple voltmeters to make comparable measurements or compare measurements with other laboratories. To achieve this precision, a sampling increment of

0.0001 ( $\Delta_{s_{xi}}$ ) volts or less is required. The precision of a measurement result is ultimately limited by the Planck constant, the minimum possible sampled increment, and is never zero.

A stochastic distribution of voltmeter measurement results occurs when applying the 1.00000 volt observable (continuous relative to the sampling increment of 0.0001 volts) to many of the same model voltmeters. Indeterminacy of the continuous observable less than the sampling increment is not identifiable. The sampling process causes each of the 100 ( $m$ ) 0.01 intervals to have an indeterminacy of  $\Delta_{s_{xi}} = -0.0001, 0.0000 (<\Delta_{s_{xi}})$  or,  $+0.0001$ . Less than  $\Delta_{s_{xi}}$  is not a zero state, but the transition between  $+\Delta_{s_{xi}}$  and  $-\Delta_{s_{xi}}$ . Therefore  $< \Delta_{s_{xi}}$  statistically occurs less often than either  $\pm\Delta_{s_{xi}}$  and the standard deviation of this sampling distribution is always  $< |\Delta_{s_{xi}}|$ .

In this sampling example, the three possible interval magnitudes (from (2)) are:  $i \pm \Delta_{s_{xi}} = i_x = 0.0099i, 0.0100i$  or  $0.0101i$ . This identifies  $3^{100}$  combinations of the 100 intervals which establish the probability of the 200 valid measurement results with values between 0.9900 to 1.0100 and within the defined  $\pm 1\%$  precision. The distribution of the  $3^{100}$  possible combinations of the 100 intervals will, as the number of voltmeter measurements increase, converge to a normal distribution (bell shaped curve) as described by the central limit theorem. Such a distribution (i.e., sampling noise) occurs in all measurement results and is sometimes identified in the literature as  $1/f$  noise, where  $f$  represents the sampling frequency in [17].

As a further example, an observable's angular rotation is determined by counting the teeth of an attached gear with  $m$  teeth per  $2\pi$  rotation. When counted by an observer, the gear teeth are assumed to be intervals of equal width and no indeterminacy appears. But a perfect measurement of angular rotation by using the gear teeth is not possible. As  $m$  becomes larger precision increases, but  $\Delta_s$  always remains just less than the wavelength of the highest frequency light reflected from a gear tooth and visible to an observer's eye.

Calibration correlates each interval of the experimental measuring apparatus to external or non-local references [18] (from a representational measurement system) for the purpose of comparison. In classical metrology, calibration is assumed to be linear (very close to correct when  $i_x \gg \Delta_{s_{xi}}$ ). Then the minimum interval variation cancels in normal distributions. As example: the 0.00 display of the voltmeter is adjusted, a known 3.00 volt *measurement standard* is applied and then a different voltmeter adjustment (calibration) changes the display to:  $300 \times 0.01 = 3.00$  volt.

The effect of both the calibration and sampling processes on a theoretical measurement result is excluded in representational measurement theory. This exclusion results in the dichotomy, presented in (4), between existing measurement theories based upon representational measurement theory (second function) and classical metrology measurement results (third function). Section 4 defines the value of the fourth function.

#### 4. SAMPLING ENTROPY

The division between discrete and continuous observables is determined by the sampling frequency,  $f$ . That is, when  $f \geq 2$  times the highest frequency of the observable, the observable's intervals are often assumed to be discrete relative to the measuring apparatus. Such a measurement assumption may be described as counting. When  $f < 2$  times the highest

frequency of the observable, the observable appears continuous relative to the measuring apparatus and measurement results often have a specified precision and accuracy.

Shannon [19] identifies this entropy difference between the discrete and continuous processes that produce a measurement result: "There is one important difference between the continuous and discrete entropies. In the discrete case the entropy measures in an absolute way the randomness of the chance variable. In the continuous case the measurement is *relative to the coordinate system*". The italics are Shannon's.

Shannon's formal development [20] provides (5), the sampling entropy due to  $\Delta s$ .

$$H(\Delta s) = -(\log_e \sqrt{2\pi e} + \log_e \sigma(\Delta s)) \quad (5)$$

As was demonstrated in Section 3,  $\sigma(\Delta s)$  is near 1 in classical measurements with normal sampling distributions, so the  $\log_e \sigma(\Delta s)$  is near zero. This makes it practical in many classical measurements to ignore the sampling effects. In quantum measurements where  $\sigma(\Delta s)$  may not be near 1, this entropy change must be treated.

## 5. RELATIONAL STANDARD DEVIATION

Metrology establishes a hierarchy of measurement references (standards) to establish accuracy, beginning with the primary standards, then intermediate standards, and finally to the calibration of a measuring apparatus [21]. When a measuring apparatus is calibrated to  $x$  (a specific standard), its accuracy is measured by a relational standard deviation, where the mean of the intervals is replaced by a standard interval magnitude,  $x$ . Based upon (1), the *relational standard deviation* is defined as:

$$\sigma(i_x) = \sigma(x + \Delta x_i) = \sqrt{\frac{1}{m} \sum_{i=1}^{i=m} (x + \Delta x_i)^2} \quad (6)$$

Considering that  $x > \Delta x_i$ , each  $\Delta x_i$  may vary, and that square roots (except of the rare perfect squares) are irrational numbers, the magnitude of (6) is almost always an irrational number, which is indeterminate. This is a formal demonstration that the calibration process, like the sampling process, is almost always indeterminate. Therefore, the magnitude of the relational standard deviation of the intervals will vary with the accuracy of calibration.

## 6. UNCERTAINTY RELATIONS

A measurement disturbance, where the first measurement of an observable instantaneously changes the measurement of a second correlated observable is well known in theory and practice [22]. The measurement disturbance appears in the uncertainty relation:

$$\sigma(i_a)\sigma(i_b) = \sigma(A)\sigma(B) \geq \frac{1}{2} \left| \int \psi^* (AB - BA) \psi dx \right| \quad (7)$$

The terms on either side of  $\geq$  in (7) are the Robertson form [23] of the HUR (where  $A$  and  $B$  need not be canonical conjugates). The left most term of (7) is from (6) and calculates the standard deviations of the intervals,  $i_a$  or  $i_b$ ; the middle term calculates the standard deviation of the sum of each observable's intervals, which is  $\sigma(A)$  or  $\sigma(B)$ . When  $m$  (magnitude of  $A$ ) and

$m'$  (magnitude of  $B$ ) are constant (i.e.,  $m$  and  $m'$  are magnitudes of fixed observables), then  $\sigma(i_a) = \sigma(mi_a) = \sigma(A)$  and  $\sigma(i_b) = \sigma(m'i_b) = \sigma(B)$ . When the observables are fixed, (7) is not correlated to  $m$  or  $m'$  as has been assumed in quantum formalisms [24].

In (7) ( $AB - BA$ ), the first term  $AB$  indicates the observable  $A$  is measured first, then  $B$  and the reverse in the second term. The magnitude of the first term is almost always different than the second, identifying the first measurement changes a second differently each time. While this does not appear logical when measuring a fixed observable, it follows directly when the relational standard deviation (6) of the set of the intervals of each observable is considered.

As the scale of an experimental measurement approaches zero,  $\Delta i_a$  and  $\Delta i_b$  (indeterminacy of  $i_a$  and  $i_b$ ) and  $i_a$  and  $i_b$  (intervals) both approach the Planck constant as a limit. Then the indeterminacy as a percentage of an interval increases in quantum scale measurements but is very small in measurement results above the quantum scale.

Based upon recent experiments and formal developments (see Appendix), (7) expands into (8). The indeterminacy relation, (8) is the product of accuracy,  $\sigma(x + \Delta i_a)$  and precision  $\sigma(\Delta s_{ai})$ , for one observable and the equivalent  $\sigma(x + \Delta i_b)$  and  $\sigma(\Delta s_{bi})$  for the other observable.  $x$  is the reference unit common to  $A$  and  $B$ .

$$[\sigma(x + \Delta a_i) + \sigma(\Delta s_{ai})][\sigma(x + \Delta b_i) + \sigma(\Delta s_{bi})] \geq \frac{1}{2} \left| \int \psi^* (AB - BA) \psi dx \right| \quad (8)$$

The product of the left side of (8):

$$\sigma(x + \Delta a_i)\sigma(x + \Delta b_i) + \sigma(x + \Delta a_i)\sigma(\Delta s_{bi}) + \sigma(\Delta s_{ai})\sigma(x + \Delta b_i) + \sigma(\Delta s_{ai})\sigma(\Delta s_{bi}) \quad (9)$$

The sum of the four variances in (9) is shown in the Appendix to be the same as the universal uncertainty relation (12). When  $\Delta s$  is assumed to be 0, the variance of (9) is  $\sigma(x + \Delta a_i)\sigma(x + \Delta b_i)$ , equal to the HUR. The sum of the four variances in (9) will rarely be equal when resampled or recalibrated. In agreement with the experimental results and theory referenced in the Appendix, the calibrations and samplings to references establish the correlation between  $A$  and  $B$ , and measurement intervals will remain within  $\pm \Delta i_x$  if the relational measurement system is not changed by time evolution, resampling or recalibration.

The terms of (9) are expanded to better identify how precision (10) and accuracy (11) appear:

$$\sigma(\Delta s_{ai}) = \sqrt{\frac{1}{m} \sum_{i=1}^{i=m} \left( \Delta s_{ai} - \frac{1}{m} \sum_{i=1}^{i=m} \Delta s_{ai} \right)^2} \quad \sigma(\Delta s_{bi}) = \sqrt{\frac{1}{m} \sum_{i=1}^{i=m} \left( \Delta s_{bi} - \frac{1}{m} \sum_{i=1}^{i=m} \Delta s_{bi} \right)^2} \quad (10)$$

Applying (10) and assuming no other interval indeterminacy than sampling, the mean of all  $\Delta s_{ai} = 0$ . Then  $\sigma(\Delta s_{ai} \text{ or } \Delta s_{bi})$  approaches  $I$ , because  $\Delta s_{ai} \text{ or } \Delta s_{bi} = 0$  or  $\pm I$  increment, and  $\Delta s_{ai} \text{ or } \Delta s_{bi} = 0$  is statistically rare. This analysis is corroborated by the experiments described in the Appendix. The standard deviations (10) represent the sampling effects related to the local intervals and do not include the effects related to the non-local reference  $x$ , shown in (11).

$$\sigma(x + \Delta a_i) = \sigma(i_a) = \sqrt{\frac{1}{m} \sum_{i=1}^{i=m} (x + \Delta a_i)^2} \quad \sigma(x + \Delta b_i) = \sigma(i_b) = \sqrt{\frac{1}{m} \sum_{i=1}^{i=m} (x + \Delta b_i)^2} \quad (11)$$

The product of the two equations in (11) is  $\sigma(ia)\sigma(ib) = \sigma(x + \Delta a_i)\sigma(x + \Delta b_i)$ , the same as the left most variance (calibration) in (9) and the same as the left most term in (7), the HUR. This is the proof that a measurement disturbance is established by calibration. This is corroborated by the experiments in the Appendix.

## 7. SUMMARY

The result of an experimental measurement of a continuous observable is the sum of a finite set of imperfect intervals. These intervals are established by sampling, which is imperfect division, and may be partially corrected by calibration. The external (from the measurement system) calibration reference and sampling frequency reference are applied in rigorous experimental measurements. Non-linear changes correlated to these external references change the entropy of the measurement system. In classical metrology, these entropy changes appear as the indeterminacy of a measurement result, termed accuracy and precision. In current quantum mechanics, these entropy changes appear as a measurement disturbance.

Since all experimental measurements are calibrated and have finite sampling increments, representational measurement theory does not represent the experimental measurement of continuous observables. Relational measurement theory, by including the external references, supports consistent measurement calculations and experimental results at all scales.

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## Appendix

Experimental verification of the universal uncertainty relation (12) is provided by neutron spin axis position experiments on a stream of neutrons (observables) [10]. This Appendix relates these experiments and the formal development [3] the experiments are based upon, to the relational measurements paper. A complete description of the experiments or the formal developments is provided in the source references. The notation in this Appendix follows the notation in each source referenced, except where noted.

### EXPERIMENTAL VERIFICATION

In the first measurement of the experiments, the measuring apparatus **A1** is detuned (de-calibrated) in a way that it projectively measures the output operator  $O_A$  instead of  $A$  thus establishing an error  $\epsilon$  in the  $A$  measurement. The subsequent measurement of the orthogonal observable  $B$  in the eigenstate of  $O_A$ , performed by measuring apparatus **A2**, virtually modifies  $B$  to be  $O_B$  from whose expectation values the disturbance  $\eta$  on  $B$  can be determined (from Fig. 5 in [10]).

The neutron spin measurement process begins with adjustments which fix at zero the azimuthal angle,  $\phi_{OA}$  which is the detuning between  $A$  and  $O_A$ .  $\phi_{OA}$  is the measure of calibration in relational measurements.  $O_A$  is a metric of time/distance relative to the fixed precession period,  $2\pi$  radians, which is the reference unit of the measuring apparatus and appears as a distance along the  $y$  axis. Referring to Fig. 6 in [10], this preparation process establishes the physical placement of DC-2 to DC-3 relative to each other and to Analyzer 1. This adjustment process correlates the physical distance on the  $y$  axis between DC-2 and DC-3 to  $n \times 2\pi$  radians. For the measurement of  $O_B$  relative to  $B$ , DC-4, Analyzer 2 and the final detector are similarly adjusted to Analyzer 1.

After the above adjustments are complete the detuning of  $O_A$  may be shown. The detuning process changes a distance on the  $y$  axis relative to the  $2\pi$  Larmor precession which is the reference. The  $2\pi$  radians represents a fixed distance on the  $y$  axis that is further divided into each detuning angle ( $\phi_{OA}$ ) shown in Fig. 8 in [10]. When DC-2/DC-3 is detuned relative to Analyzer 1 and Analyzer 2,  $O_A$  is de-calibrated from  $A$ ,  $O_B$  is de-calibrated from  $B$ , and both  $\epsilon(A)$  and  $\eta(B)$  change as shown in Fig. 8. The two orthogonal dimensions are not related by a Fourier

transform. Thus the measurement results of  $A$  and  $B$  are only correlated by their common reference. Without the additional correlation of a Fourier transform, the effect of de-calibration on the measurement results (Fig. 10 in [10]) follows Ozawa's theory [3] closely.

When  $\phi_{OA}$  is zero,  $O_A=A$ . This is the discrete case where the measurement interval and the reference interval are equal and there is no calibration effect. As this experiment has reasonable precision ( $\sim 1.5^\circ$ , stated in page 8 in [10]) the sampling effects,  $\sigma(A)$  and  $\sigma(B)$  are close to  $I$  increment which verifies the analysis in Section 3.

## FORMAL DEVELOPMENT

The experiments above verify the universal uncertainty relation, equation (266) in [3], as:

$$\epsilon(A, \rho)\eta(B, \rho) + \epsilon(A, \rho)\sigma(B, \rho) + \sigma(A, \rho)\eta(B, \rho) \geq \frac{1}{2} |Tr([A, B]\rho)| \quad (12)$$

Equation (12) is equivalent to the indeterminacy relation shown in (9), the expanded version of (8). In (12)  $\epsilon(A)$  is a disturbance of  $A$  caused by a measurement and  $\eta(B)$  is the noise of  $B$  that occurs due to this disturbance.  $\rho$  (applying the relational measurement vocabulary) represents a normalized possible measurement interval. In (12) the standard deviations shown as  $\sigma(A, \rho)$  and  $\sigma(B, \rho)$  represent the sampling effects the same as  $\sigma(A)$  and  $\sigma(B)$  in [10]. When comparing the left side of (12) to (9), (12) must be slightly modified. This is developed below.

In (12)  $\epsilon(A, \rho)$  and  $\eta(B, \rho)$  are two root mean squared values each calibrated to  $x$  (termed a true value  $[a]$  in [3] page 17 bottom of the right column). Thus  $\epsilon(A, \rho)$  and  $\eta(B, \rho)$  are equal to the relational standard deviations  $\sigma(x + \Delta a_i)$  and  $\sigma(x + \Delta b_i)$  in (11). The next two product terms identify the variance due to the sampling effects on these two relational standard deviations.

Note that a fourth term,  $\sigma(A, \rho)\sigma(B, \rho)$ , appears mathematically consistent in (12). Modifying (12) to include this fourth term does not change the inequality of (12). This modification is supported by:  $A$  and  $B$  are correlated by sampling to a common reference (time) in the neutron spin example (not uncorrelated as expressed in [3] (235)). Therefore the product of the "pre-measurement uncertainties" ([3] page 25 section C) should be included in (12). And, the test results in [10] show that both sampling standard deviations are slightly less than one, therefore their product is close to one, not zero.

The point of modifying (12) to include this fourth term is to factor the modified (12) into (8). Equation (8) identifies that the first standard deviation term of each observable is a calibration process and the second is a sampling process. The experimental results can ignore this fourth term because the sampling entropy (5), a constant plus the log of close to one (which is close to zero), so the fourth term has little effect on the experimental results. But it does effect the understanding of indeterminacy.

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